J. Fluid Mech. (1977), *vol.* 79, *part* 4, *pp.* 657–668
Printed in Great Britain

A note on quasi-geostrophic flow over topography in bounded basin

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(Received 10 February 1976 and in revised form 14 September 1976)

We consider the flow of rapidly rotating fluid over topography in a circular basin. The equations of motion (here the inviscid quasi-geostrophic vorticity equations) can be integrated exactly for certain zonally averaged currents. The assumption of the existence of a specified zonal current is equivalent to the assumption of no upstream influence in the unbounded case. It is unlikely that such solutions can be realized in experiments with real fluids for the presence of viscosity, however small, causes 'zonal influence' *independent* of the magnitude of the viscosity at times larger than the spin-up time. For times smaller than the spin-up time decaying transients can cause zonal influence which increases in magnitude with decreasing viscosity.

1. Introduction

The slow flow of a rapidly rotating fluid over topography has been the subject of much recent theoretical discussion. It is well known that surface protuberances or depressions of sufficient height can block the flow; streamlines originating upstream of the obstacle go around rather than over it. The extent to which such blocking occurs in a fluid with non-zero stratification was discussed by Hide (1971) and Hogg (1973). In these and other theoretical studies, solutions are obtained for steady flow by noting that the potential vorticity (relative vorticity plus suitably non-dimensionalized depth) is constant along a streamline. If each streamline is traced far upstream of the obstacle, where the potential vorticity is known, a simple linear problem usually results for the determination of the exact stream field everywhere. That the potential vorticity is known upstream constitutes what has become known as Long's hypothesis or the condition of no upstream influence. In the problem of stratified non-rotating flow of obstacles, McIntyre (1972) has demonstrated that upstream influence occurs for inviscid transient motion bounded above and below. In horizontally unbounded, weakly viscous flows he finds that the columnar disturbances responsible for the upstream influence vanish with the viscosity when either the upper or the lower boundary is 'no-slip'. We show below that this does not happen for weakly viscous, rapidly rotating flows over topography in a cyclic container and that

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one must therefore question the physical realizability of solutions obtained using Long's hypothesis in such situations.

From an experimentalist's point of view, the use of extremely long tanks to study the delicate question of the validity of solutions based on zero upstream influence constitutes a difficult and often impractical problem. Thus many laboratory studies of such flows have been done in rotating cylinders of various types with either a body or 'mountain' moving through the fluid, or a stationary bump or field of bumps on the bottom of the container with motion driven by a differentially rotating horizontal disk in contact with the top of the fluid. An example of the former is Davies's (1972) experiments on motion of a continuously stratified fluid past a sphere and an example of the latter is Hart's (1972) study of baroclinic instability in a two-layer fluid over various bottom topographies. In a cylinder where the variable topography is not confined to a very small area of the bottom, it is no longer clear what is meant by 'upstream' since the container is cyclic.

We shall show below that the specification of an upstream flow in an infinite channel, which yields a simple linear problem for the determination of an exact solution of the nonlinear potential-vorticity equation, is equivalent to specifying certain zonally averaged flows $v_0(r)$ in the cylindrically cyclic geometry. The problem of upstream influence is then restated as a question of whether or not the zonal flow needed to obtain an exact solution valid for all topographic scales (consistent with the quasi-geostrophic equations we shall be using, but still large enough to drive a strongly nonlinear response) is 'influenced' by the θ-dependent motions excited as fluid flows over the bump. The problem breaks up into two time regions. For those times substantially less than the spin-up time, transients caused by the introduction of the obstacle can, except for a few restrictive cases, cause permanent or growing distortions of the zonal current. Because of the difficulty of comparing time-dependent flow theory with experiment, since the details of the transients depend on how the obstacle is introduced or on how the current is turned on, we are more interested in cases where a small viscosity is present. Then, for times much greater than the spin-up time $\tau^* = H/(\nu\Omega)^{\frac{1}{2}}$ the transient effects will have disappeared and there will be no zonal influence apart from that of the steady motion itself. At these times a zonal influence entirely independent of viscosity will persist.

2. Inviscid theory

There are two main points we wish to demonstrate in this note. We shall first show that when the viscosity is zero the steady solution of a linearized potential-vorticity equation based on small topography is an exact solution for a certain fairly broad class of zonal flows. These simply calculable solutions can then be used in situations where the topography is large. The second, and major, point of this study is to examine the effects on these exact solutions of small viscosity, which leads to the presence of quasi-horizontal Ekman layers on rigid boundaries. It will be shown that any viscosity, no matter how small, will generate corrections to the linearized solutions, the largest of which are independent of the magnitude

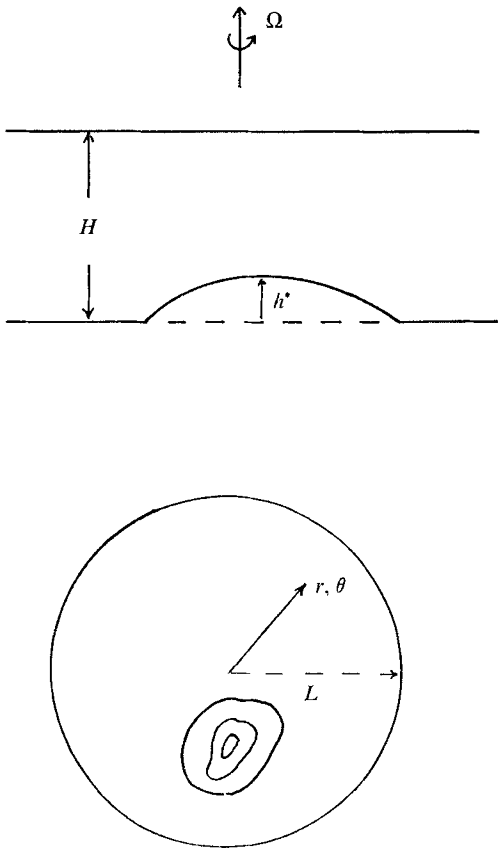

FIGURE 1. Geometry for the study of slow flows over topography in a rotating fluid.

of the viscosity. Hence the inviscid dynamics are singular in the sense that the presence of very small viscosity can cause substantial modifications to the inviscid solutions.

These results are most easily demonstrated in the context of a one-layer fluid model. They can easily be extended to multi-layer fluids which have small friction at immiscible interfaces between isopycnal layers, although the algebra becomes rather cumbersome. The basic physical process which causes the zonal correction is not significantly affected by stratification provided it is not so strong as to choke off the Ekman suction velocity. Thus we consider the simple configuration shown in figure 1. A layer of homogeneous incompressible fluid is contained between a horizontal lid and a lower boundary which contains some general topography

$$h^* = \delta h . h(r, \theta),$$

where $h(r, \theta)$ is of order one. The mean depth of the fluid is H and its viscosity is ν. Velocities have scale U, which is a measure of the speed of the flow incident on the topography. It is presumed that the system is rotating rapidly at a rate Ω, so that the Rossby number $R_0 = U/2\Omega L$ based on the horizontal scale L of either the basin or the mountain range is very small. Thus the velocities are primarily horizontal, independent of z and are in geostrophic balance:

$$v = \partial P/\partial r, \quad u = -r^{-1}\partial P/\partial \theta, \qquad (2.1), (2.2)$$

where P is the dynamic pressure scaled by $2\rho U \Omega L$ and u and v are the radial and azimuthal velocity components scaled by U.

The governing prognostic equation, accurate to terms of the order of R_0 or $\delta h/H$, is the well-known quasi-geostrophic vorticity equation. This expresses a balance between vorticity fluctuations following fluid columns and generation due to stretching of the planetary vorticity 2Ω by flow over topography or by suction into thin Ekman layers at the boundaries. Lateral friction is only important on horizontal scales much smaller than those of interest here and is neglected; $\nu/2\Omega L^2 \ll R_0$.

Let $\omega \equiv \nabla^2 P$ be the relative vorticity. Then the vorticity equation is

$$\partial\omega/\partial t + J(P,\omega) = -\alpha J(P,h) + \chi(\hat{\mathbf{k}}\,.\,\mathrm{curl}\,\mathbf{v}|_w - \omega). \qquad (2.3)$$

The two parameters are

$$\alpha \equiv \delta h/HR_0 \lesssim O(1) \qquad (2.4)$$

and

$$\chi \equiv 2^{\frac{1}{2}}E^{\frac{1}{2}}/R_0 \lesssim O(1), \qquad (2.5)$$

where E is the Ekman number $\nu/2\Omega H^2$. It is assumed that, if non-zero, both α and χ are much bigger than either R_0 or $\delta h/H$, so that topography or bottom friction will dominate ageostrophic effects.

We thus take (2.3) to be our governing equation. Because of the Jacobian operator

$$J(P,\omega) \equiv \frac{1}{r}\left(\frac{\partial P}{\partial r}\frac{\partial \omega}{\partial \theta} - \frac{\partial P}{\partial \theta}\frac{\partial \omega}{\partial r}\right)$$

this equation is nonlinear. We have included the possibility of driving the flow in the basin by moving the upper lid at velocity $\mathbf{v}|_w$. This will be useful below when we look for steady solutions when weak viscosity is present. The speed of the driving lid has scale U at a radius L.

Let us begin by looking for steady inviscid solutions of (2.3). It is convenient to write the non-dimensional topography $h(r,\theta)$ as the sum of a cyclic and an azimuthally averaged part:

$$h = h'(r,\theta) + \bar{h}(r),$$

where the overbar denotes a zonal average generally defined by

$$\bar{f} = \frac{1}{2\pi}\int_0^{2\pi} f(r,\theta)\,d\theta$$

for any variable f. Under certain circumstances there are exact, steady, inviscid solutions of the nonlinear equation (2.3) of the form

$$P = P_0(r) + \alpha P_1(r,\theta)$$

for which the differential equation for P_1 is linear.

In the absence of topography ($\alpha \to 0$) we assume that a known zonal flow is present. Any $P_0(r)$ distribution is allowed since $J(P_0, \nabla^2 P_0) \equiv 0$. Suppose that the modification P_1 due to topography is described by the simplified equation

$$\frac{1}{r}\left[\frac{\partial P_0}{\partial r}\frac{\partial \omega_1}{\partial \theta} - \left(\frac{\partial \omega_0}{\partial r} + \alpha\frac{\partial \bar{h}}{\partial r}\right)\frac{\partial P_1}{\partial \theta}\right] = -\frac{1}{r}\frac{\partial P_0}{\partial r}\frac{\partial h'}{\partial \theta}.$$

This is obtained from (2.3) by neglecting time variations, viscous effects, the nonlinear self-interaction of the P_1 field and the interaction between the P_1 field and the cyclic part h' of the topography. It is linear in P_1, and the solution $P = P_0 + \alpha P_1$ will be called the linearized solution.

The above equation can be written as

$$\frac{\partial \omega_1}{\partial \theta} - g(r)\frac{\partial P_1}{\partial \theta} = -\frac{\partial h'}{\partial \theta}, \tag{2.6}$$

where

$$g \equiv \partial \omega_0/\partial P_0 + \alpha \partial \bar{h}/\partial P_0. \tag{2.7}$$

The quantity g is a measure of the zonal potential-vorticity fluctuation between zonal streamlines. The general solution of (2.6) is

$$P_1 = p_1(r, \theta) + F_1(r),$$

where

$$\nabla^2 p_1 = g p_1 - h'. \tag{2.8}$$

With impermeable boundaries (2.8) is solved with $p_1(1, \theta) = 0$.

In the inviscid theory the free solution is not uniquely determined and must be specified on physical grounds. The P_1 field is generated by an interaction between the zonal flow and the cyclic topography. There is no mechanism (in view of our neglect of self-interactions) by which a zonal component of P_1 may occur. Thus, since p_1 is cyclic, F_1 must be zero.

If we want $P = P_0 + \alpha p_1$ to be the total exact solution we must require that the nonlinear terms omitted from (2.6) vanish. These terms are

$$\alpha^2 J(P_1, \nabla^2 P_1 + h') = -\frac{\alpha^2}{2r}\frac{\partial g}{\partial r}\frac{\partial p_1^2}{\partial \theta}. \tag{2.9}$$

Thus our linearized solution is an exact inviscid steady solution if g is a constant. The zonal component of the potential vorticity must be a linear function of P_0. For general α this condition requires that both $\partial \omega_0/\partial P_0$ and $\partial \bar{h}/\partial P_0$ be constants. For particular values of α there may be situations where $\partial \omega_0/\partial P_0 = -\alpha \partial \bar{h}/\partial P_0$ at all radii. These are constraints on the zonal flow and the zonal component of the topography which must be satisfied if the easily calculated $P_0 + \alpha p_1$ solution is to satisfy the full inviscid equation.

Let us note the relation of these exact linearized solutions to those obtained directly. It is usually argued that $\nabla^2 P + \alpha h = G(P)$ for steady inviscid nonlinear solutions of the vorticity equation. G is a regular function of P. Upstream of the topography, it is assumed that $P = P_0$ and $h = 0$. Thus $G(P_0) = \nabla^2 P_0$.

$$G(P) = \text{constant} \times P \quad \text{if} \quad \partial \omega_0/\partial P_0 = \text{constant},$$

and the resulting equation for P is linear and identical to (2.8) if $\bar{h} = 0$. In the cyclic geometry the condition that h be zero outside a finite region is not always enough to give a linear equation for P, as it is in an infinite channel.

Suppose that we wish to realize a situation where $g' = 0$. If we generate a zonal flow at an initial time with these properties, we must require that the mountain-induced flow causes no zonal influence which may alter P_0. Referring to (2.9) it is seen that there is no mechanism for generating a higher-order zonal flow for any $g(r)$. The inviscid flow cannot produce zonal influence. It must come from either the transient flows during the setting-up of the experiment or from the action of viscosity in the steady state. The zonal influence during the transient stage of the flow depends on the details of the introduction of the topography or on the setting-up of the zonal current. There will generally be a non-zero order-α^2 rectification of the transient field which produces a zonal influence which grows linearly with time in an inviscid fluid. In the appendix it is shown how the transients affect the zonal current. If the fluid is inviscid a steady state is never reached. This presents serious difficulties in an experimental test of the steady theory. However a real fluid possesses some viscosity. If viscous effects are small ($E^{\frac{1}{2}}/R_0 \ll 1$) the transients decay like $\exp(-E^{\frac{1}{2}}/R_0 t)$. The zonal correction due to the presence of the transients then goes like

$$(f(r)/\chi)\,[e^{-2\chi t} - e^{-\chi t}].$$

Thus the zonal correction peaks at $t = \ln 2/\chi$, about one spin-up time. Its peak value is proportional to $\chi^{-1} = R_0/2^{\frac{1}{2}}E^{\frac{1}{2}}$ and so increases with decreasing viscosity. None the less, at times substantially greater than the spin-up time $t_s^* = H/(2\Omega\nu)^{\frac{1}{2}}$, the zonal influence of the transients and the transients themselves have negligible amplitudes. Because of the difficulty of correlating experiment and theory in the transient region (a difficulty which would seem to increase with decreasing E) one perhaps would want to wait until viscosity had damped out all transients and concentrate on the steady flow structure for $t^* \gg t_s^*$.

3. Steady slightly viscous motion

We now wish to examine the role of viscosity in generating zonal corrections to the mean axisymmetric flow which would occur in the absence of the mountain. In the example computed here we suppose that $E^{\frac{1}{2}}/R_0$ is of order α^2, although the results can be generalized for $E^{\frac{1}{2}}/R_0 \propto \alpha^n$ for any n, $n = 1$ being the simplest. The governing equation is again (2.3), i.e.

$$J(P, \nabla^2 P + \alpha h) = -\chi(\nabla^2 P - 1), \tag{3.1}$$

where the forcing on the right-hand side represents that in a typical laboratory situation where the fluid is driven from above by a differentially rotating lid. In the absence of any mountains, the interior solution would be solid-body rotation at a rate half that of the driving frequency. Consider an expansion in powers of α. We write

$$P = P_0 + \alpha P_1 + \alpha^2 P_2 + \dots.$$

At order α^0 we have

$$J(P_0, \nabla^2 P_0) = 0$$

and we choose $P_0 = P_0(r)$ to be consistent *a posteriori* with a continuous parameter space limit to $\alpha = 0$, where only a zonal flow occurs. At $O(\alpha)$,

$$J(P_0, \nabla^2 P_1 + h) + J(P_1, \nabla^2 P_0) = 0$$

so

$$P_1 = p_1 + F_1$$

as before, where

$$\nabla^2 p_1 = p_1 \partial\omega_0/\partial P_0 - h, \quad p_1(1,\theta) = 0. \tag{3.2}$$

Let $\epsilon \equiv \chi/\alpha^2$. Then at $O(\alpha^2)$

$$\frac{1}{r}\left[\frac{\partial P_0}{\partial r}\frac{\partial \nabla^2 P_2}{\partial \theta} - \frac{\partial \nabla^2 P_0}{\partial r}\frac{\partial P_2}{\partial \theta}\right]$$
$$= -\epsilon(\nabla^2 P_0 - 1) + \frac{1}{r}\frac{\partial p_1}{\partial \theta}\frac{\partial \nabla^2 F_1}{\partial r} + \frac{1}{2r}\frac{\partial^2 \omega_0}{\partial r \partial P_0}\frac{\partial p_1^2}{\partial \theta} - \frac{1}{r}\frac{\partial F_1}{\partial r}\frac{\partial \omega_0}{\partial P_0}\frac{\partial p_1}{\partial \theta}. \tag{3.3}$$

Averaging (3.3) gives an equation for P_0, namely

$$\nabla^2 P_0 = 1,$$

or

$$P_0 = \tfrac{1}{4}r^2.$$

This is the basic forced axisymmetric flow in the absence of mountains, and satisfies our general inviscid conditions if $\bar{h} \propto r^2$. The solution of (3.3) is

$$P_2 = p_2 + F_2(r),$$

where

$$\nabla^2 p_2 = \frac{2}{r}\frac{\partial}{\partial r}(\nabla^2 F_1)\, p_1(r,\theta). \tag{3.4}$$

F_1 and F_2 are as yet arbitrary. In this viscous theory they are determined from higher-order equations.

At $O(\alpha^3)$

$$\tfrac{1}{2}\partial(\nabla^2 P_3)/\partial\theta = -\epsilon\nabla^2(F_1 + p_1) - J(P_1, \nabla^2 p_2 + \nabla^2 F_2) - J(p_2, \nabla^2 F_1). \tag{3.5}$$

Averaging in azimuth and using (3.4) we find that all terms are zero except that related to viscosity. Thus $F_1 = -\bar{p}_1$. The full solution up to order α is obtained by solving (3.2) with $h - \bar{h}$ replacing h and P_1 replacing p_1. Viscosity does not allow *any* zonally averaged currents at order α. Thus, up to order α, the solution *with* viscosity is identical to the inviscid one.

We can write the $O(\alpha^3)$ solution as

$$P_3 = p_3 + \tilde{p}_3 + F_3(r),$$

where

$$\nabla^2 \tilde{p}_3 = -\frac{4}{r^2}\frac{\partial F_1}{\partial r}\frac{\partial(\nabla^2 F_1)}{\partial r} p_1 + \frac{2}{r}\frac{\partial(\nabla^2 F_1)}{\partial r} p_2 + \frac{2}{r}\frac{\partial(\nabla^2 F_2)}{\partial r} p_1 + \frac{2p_1^2}{r}\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial(\nabla^2 F_1)}{\partial r}\right) \tag{3.6}$$

and

$$\partial(\nabla^2 p_3)/\partial\theta = -2\epsilon\nabla^2(p_1 + F_1). \tag{3.7}$$

At order α^4 the zonal influence F_2 is determined. We have

$$\frac{1}{2}\frac{\partial}{\partial\theta}\nabla^2 P_4 = -\epsilon\nabla^2(p_2 + F_2) - J(p_2 + F_2, \nabla^2(p_2 + F_2))$$
$$- J(p_3 + \tilde{p}_3, \nabla^2 F_1) - J(p_1 + F_1, \nabla^2(p_3 + \tilde{p}_3 + F_3)). \tag{3.8}$$

We average over θ and note that $\overline{\nabla^2 (p_2 + F_2)} = \nabla^2 \overline{P}_2$. Then we use (3.4), (3.6) and (3.7) to reduce the Jacobian terms that survive after the averaging has eliminated all those terms which have functions solely of r as one argument. This yields the very simple result

$$\nabla^2 \overline{P}_2 = -\epsilon^{-1}\overline{J(P_1, \nabla^2 p_3)} = \frac{1}{\pi r}\int_0^{2\pi} \frac{\partial}{\partial r}[P_1 \nabla^2 P_1]\,d\theta.$$

The zonal influence $\bar{v}_2 = \partial \overline{P}_2/\partial r$ is given by

$$\bar{v}_2 = \frac{-1}{\pi r}\int_0^{2\pi} P_1(h-\bar{h})\,d\theta = \frac{-1}{\pi r}\int_0^{2\pi} P_1 h d\theta. \tag{3.9}$$

From (3.7) we see that p_3 is out of phase with the primary solution P_1. This introduces a Reynolds stress of order $\alpha^2 E^{\frac{1}{2}}/R_0$ which has a non-zero zonal average and is divergent in general. This zonally averaged Reynolds stress in turn forces a zonal flow which is independent of the magnitude of the viscosity, because both the phase shift which generates this zonal flow and its damping are linear in χ. If (3.9) is non-zero an order-α^2 zonal correction occurs which will cause harmonics to propagate through the system and render the linear solution non-exact. That is, even if $\partial\omega_0/\partial P_0$ and $\partial\bar{h}/\partial P_0$ are constants, the weakly viscous system has corrections of order α^2 and higher. A careful inspection of the nature of the expansion shows that the first zonal correction will always be of order α^2, regardless of the order of $E^{\frac{1}{2}}/R_0$. One might expect this from the $E^{\frac{1}{2}}/R_0$ independence of $\bar{v}_2$. Formally this behaviour comes about because $\bar{v}_2$ is determined by balancing its viscous damping term, which is of order $\alpha^2 E^{\frac{1}{2}}/R_0$, with a Reynolds-stress divergence involving the product of p_1 gradients (order α) and the first viscous correction to p_1 (order $\alpha E^{\frac{1}{2}}/R_0$).

In the past the steady inviscid solutions have been used to investigate certain qualitative features such as blocking and flow reversal. The existence of such features usually results in having to specify a substantial value of α, which depends on the details of h and P_0 but is usually order one or greater. As long as the linear solution is exact it is permissible to let α take on large values. However, we have shown that viscosity always causes zonal influence of order α^2 and higher. The dominant correction as α is increased from small values is independent of viscosity. The series solution presented in this section indicates that zonal influence will occur in the steady state. Because the radius of convergence of the series is not readily ascertained, and because higher-order terms are not easily calculated, its use in obtaining the solution at large α is limited. However, it does demonstrate the origin of the zonal correction.

The simplest example illustrating the generation of zonal influence is obtained for solid-body basic rotation over a slope:

$$P_0 = \tfrac{1}{4}r^2, \quad h = r\cos\theta.$$

Then $\bar{h} = 0$, $\partial\omega_0/\partial P_0 = 0$ and

$$P = P_0 + \tfrac{1}{8}\alpha(r - r^3)\cos\theta.$$

This is then an exact inviscid solution of (2.3). It applies at all α, and for example

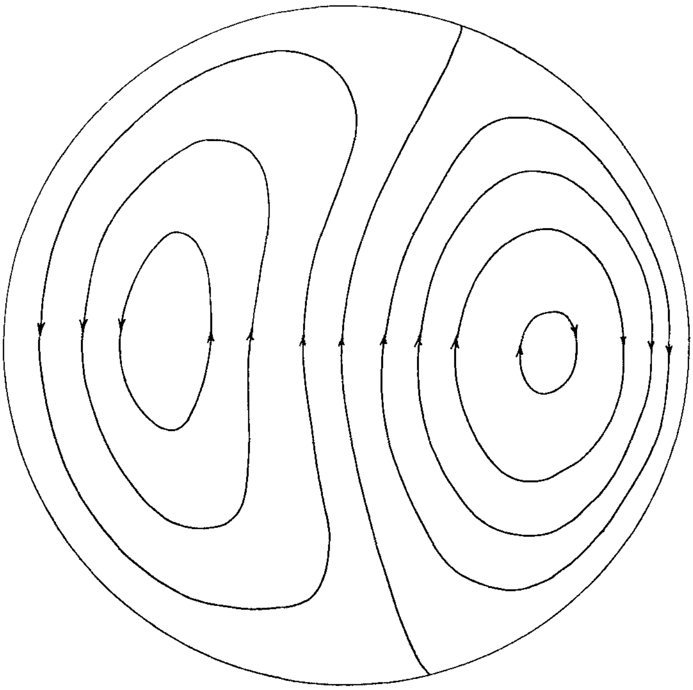

FIGURE 2. Streamline (geostrophic pressure) pattern for flow over a half-slope. The basic state is cyclonic solid-body rotation. The P_1 field is contoured here with contour interval $2{\cdot}5 \times 10^{-3}$.

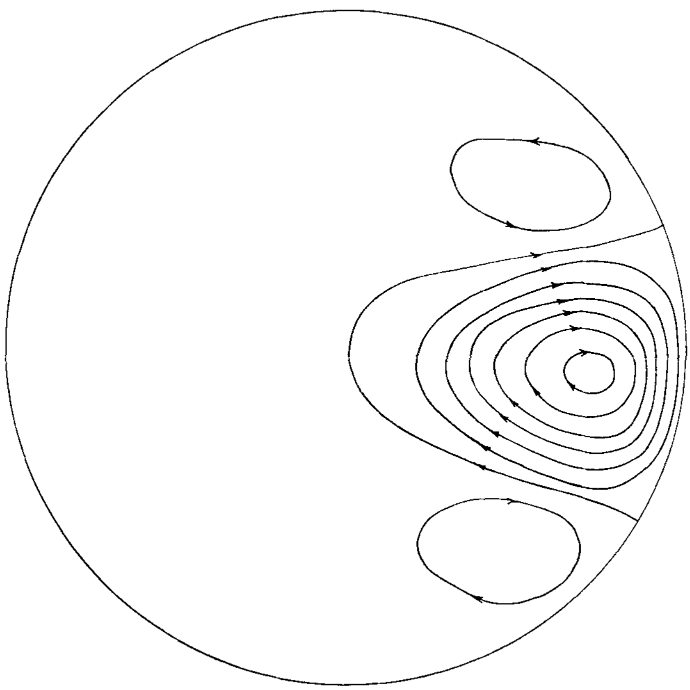

FIGURE 3. Contours of P_1 for a ridge. The contour interval is $2{\cdot}5 \times 10^{-3}$.

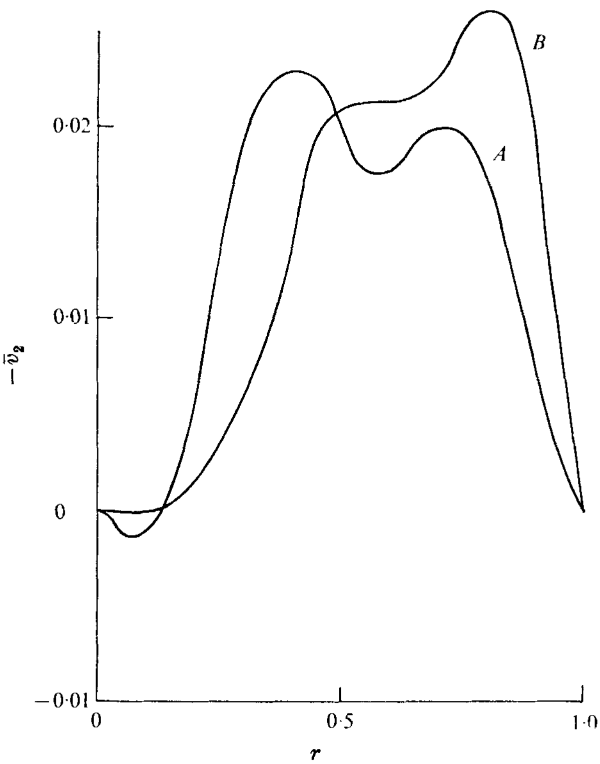

FIGURE 4. The zonal-influence function. This figure shows the θ-independent zonal correction forced by the horizontal Reynolds stresses set up by a mountain vortex (see figures 2 and 3). Curve A is for a half-slope and B is for a ridge.

predicts a counterflow near $r = 1$, $\theta = 0$ when $\alpha > 2$. The zonal influence, from (3.9), is

$$\bar{v}_2 = \tfrac{1}{8}(r^3 - r).$$

It is opposite in sign to the basic state v_0, and represents the retardation of the basic flow caused by the presence of the mountain. The zonal correction becomes significant long before α reaches 2, and its presence suggests that it is inappropriate to predict a flow reversal from the inviscid theory regardless of how small the viscosity may be.

Zonal influence will occur for flow over general topographies since there will always be a non-zero correlation between P_1 and h'.† Anticyclonic vortices (high pressure) will occur over rises and cyclonic vortices (low pressure) will occur over depressions. Figures 2–4 show computed stream fields and zonal influence $\bar{v}_2$ for solid-body rotation ($P_0 = \frac{1}{4}r^2$) over

(i) a half-slope $h = \begin{cases} r\cos\theta & \text{for} \quad |\theta| \leqslant \frac{1}{2}\pi, \\ 0 & \text{otherwise,} \end{cases}$

† From (3.9) it follows that $\int_0^1 \bar{v}_2 r^2 dr < 0$ and hence that $\bar{v}_2 \neq 0$ for some r.

and (ii) a polar ridge $h = \begin{cases} r(1-16\theta^2/\pi^2) & \text{for} \quad |\theta| \leqslant \frac{1}{4}\pi. \\ 0 & \text{otherwise,} \end{cases}$

The stream fields demonstrate how a mountain vortex is generated directly over the topography. The zonal correction is almost everywhere opposite in sign to the basic flow, and is slightly larger for the polar ridge, which has larger slopes and hence excites a stronger p_1 response.

4. Conclusions

We have shown that the steady inviscid finite-amplitude quasi-geostrophic flow of a cylindrically bounded fluid over an obstacle is governed by a simple linear equation when the zonal potential vorticity is a linear function of the zonal stream function. The presence of viscosity generates a viscosity-independent order-α^2 correction to an initial zonal current which satisfies the criterion for exactness of the linear solution. This result suggests that in a real fluid, even when the viscosity is very small, the exact linear inviscid solution is inaccurate at finite amplitude.

The author would like to express his appreciation to the referees, whose comments helped to improve the original manuscript. This research was supported by the National Science Foundation, grant DES 74-14356.

Appendix

To investigate the nature of the transients in a problem where the topography is switched on at time $t = 0$, consider the time-dependent vorticity equation. The governing equation for the linear $O(\alpha)$ response is

$$\left[\frac{\partial}{\partial t} + \chi + \frac{1}{r}\frac{\partial P_0}{\partial r}\frac{\partial}{\partial \theta}\right]\omega_1 - \frac{1}{r}\frac{\partial P_1}{\partial \theta}\frac{\partial \nabla^2 P_0}{\partial r} = -\frac{1}{r}\frac{\partial P_0}{\partial r}\sum_{n,m} inh_{mn} J_n(l_{mn} r)\, e^{in\theta} H(t),$$

where $H(t)$ is the Heaviside step function. For no motion at $t = 0$ the transient part of the solution is (with $P_0 = \frac{1}{4}r^2$)

$$P_1 = \operatorname{Re} \sum_{n,m} \frac{-h_{mn} J_n(l_{mn} r)}{l_{mn}^2} \exp\{in(\theta - \tfrac{1}{2}t + i\chi t/n)\}. \tag{A 1}$$

Is there any rectification which will leave an impression on the zonal flow? Does $J(P_1, \omega_1)$ have a non-zero average which is quasi-steady for small χ? Let

$$P_{1n} = \sum_m \frac{-h_{mn} J_n(l_{mn} r)}{l_{mn}^2}, \quad W_{1n} = \sum_m + h_{mn} J_n(l_{mn} r).$$

Then the equation for the zonally averaged order-α^2 pressure $\phi(r, t)$ is

$$\left(\frac{\partial}{\partial t} + \chi\right)\nabla^2\phi = \frac{1}{r}\sum_{n=1}^{\infty} n \operatorname{Im}\left\{\frac{\partial}{\partial r}(P_n^* W_n)\right\} e^{-2\chi t}. \tag{A 2}$$

If $E^{\frac{1}{2}} = 0$ (no viscosity), a permanent growing zonal correction will exist unless h_{mn} is real or unless h is unimodal in m for each n. Then there will be no $O(\alpha^2)$ zonal correction, although there may still be zonal influence at $O(\alpha^4)$.

If viscosity is present we can solve (A 2) for the $O(\alpha^2)$ correction, assuming that h_{mn} does not satisfy any of the previous requirements (i.e. we assume that the topographic contour lines are tilted, e.g. not purely radial). We can solve for $\bar{v}_2 = \partial\phi/\partial r$. Let

$$\bar{v}_2 = B(t) \sum_{n=1}^{\infty} A_n(r),$$

where

$$A_n = -n \operatorname{Im}(P_n^* W_n)/r, \quad (\partial/\partial t + \chi) B = -e^{-2\chi t}.$$

Subject to $\bar{v}_2 = 0$ at $t = 0$, we find

$$B = \chi^{-1}[e^{-2\chi t} - e^{-\chi t}].$$

REFERENCES

DAVIES, P. A. 1972 Experiments on Taylor columns in rotating stratified fluids. *J. Fluid Mech.* **54**, 691–717.

HART, J. E. 1972 A laboratory study of baroclinic instability. *Geophys. Fluid Dyn.* **3**, 181–211.

HIDE, R. 1971 On geostrophic motion of a non-homogeneous fluid. *J. Fluid Mech.* **49**, 745–751.

HOGG, N. G. 1973 On the stratified Taylor column. *J. Fluid Mech.* **58**, 517–537.

MCINTYRE, M. E. 1972 On Long's hypothesis of no upstream influence in a uniformly stratified or rotating fluid. *J. Fluid Mech.* **52**, 209–243.